Second semestral examination 2016
Algebraic geometry
B.Math.Hons.IIIrd year
Instructor — B.Sury

Q 1. (4+6 marks)
(a) In $\mathbf{A}^2_{\mathbf{C}}$, determine $V(f,g)$ where $f = X^2 - Y^2 + 1, g = X^2 - XY - 1$.
(b) If $V \subseteq \mathbf{A}^2_{\mathbf{C}}$ is closed, and if $s \neq t$ are in $\mathbf{A}^2_{\mathbf{C}} - V$, show that there exists $f \in I(V)$ such that $f(s) \neq 0 \neq f(t)$.
Hint: Show how one may choose $f_s \in I(V)$, $f_s \notin I(V \cup \{s\})$ etc. and that one of f_u, f_v or $f_s + f_t$ does the job.

Q 2.
Let k be any field and $A = k[X,Y,Z]/(XZ - Y^2)$. Prove that the image $P = (x,y)$ in A of the ideal (X,Y) is prime and that P^3 is not a primary ideal.

OR

Prove that the zero ideal in $C[0,1]$ does not admit a primary decomposition.
Hint: If $P \in Ass(C[0,1])$, then show that there exists $f \in C[0,1]$ such that $P = ann(f)$ and that there exists at most one point $a \in [0,1]$ where f is non-zero.

Q 3. (3+3+4 marks)
Let k be algebraically closed and let $V \subset \mathbf{A}^n_k$ be an irreducible affine variety. Let U be a non-empty Zariski-open subset of V.
(a) Show that U ia dense in V.
(b) Let $\phi : U \to k$ be a function such that for each $x \in U$, there exists an open $U_x \subset U$ and polynomials $f,g \in K[X_1,\cdots,X_n]$ with $g(y) \neq 0$ and $\phi(y) = \frac{f(y)}{g(y)}$ for all $y \in U_x$. Prove that for any other choice $x', U_{x'}$ and f', g', the rational functions f/g and f'/g' are equal in the function field $K(V)$.
(c) If $f \in k[X]$, show that $V(Y^2 - f(X)) \subset \mathbf{A}^2_k$ is irreducible if and only if f is not a square.

Q 4.
Describe the Segre embedding of $\mathbf{P}^m \times \mathbf{P}^n$ in $\mathbf{P}^{mn+m+n}$ and prove that the image is a projective variety.

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Q 5. (3+3+4 marks)
Let k be algebraically closed.
(a) Let $f = X^3 - Y^2, g = Y^3 - X^2 \in k[X, Y]$. Determine the intersection number $I((0,0), f, g)$.
(b) If $f, g \in k[X, Y]$ have no common factors, prove that $V(f, g)$ is finite.
(c) If $f, g \in k[X, Y]$ have no common factors, prove that $k[X, Y]/(f, g)$ is a finite-dimensional k-vector space. Using this, or otherwise, show that this ring has only finitely maximal ideals.